Geometric Group Theory Survey

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Contents

Introduction

Geometric group theory is an area of mathematics that I find incredibly fascinating. It takes a leap away from the solid ground of the world of manifolds, floating in a liberated and rarefied abstraction- mysteriously extending the truths and messages foreshadowed by low-dimensional topology. It also enriches the landscape of low-dimensional topology- its interactions, ramifications and revelations have led to the development of many substantial techniques and explorations. Some of these in particular have resolved the last of Thurston's remaining open questions, marking an end to a monumental era.

Geometric group theory brings to fundamental mathematical notions such as distance, curvature and equivalence- revitalising perspectives, eloquently distilling fundamental principles, in the spirit of Euclid and the world of classical geometry. From a soaring height, the notion of quasi-isometry allows us to understand that objects previously considered distinct are in many ways indistinguishable. Properties that are invariant under quasi-isometry are coveted as steadfast pillars for what they illuminate on the nature of large-scale geometry.

Geometric group theory introduces us to and facilitates an interpretation of many remarkable families of mathematical objects- for example, $CAT(0)$-cube complexes, $CAT(0)$-spaces, buildings, $\mathbb{R}$-trees, median spaces, Coxeter groups, mapping class groups and so on. Each class possesses a rich theory, connecting multiple fields and delivering us beautiful insights.

Glimmers of a coalescence between geometry and group theory began emerging in the late 19^{th} century with the *Erlangen program*- Felix Klein's endeavour to understand Euclidean, hyperbolic and projective geometries by means of group theory as a guiding light. The development of combinatorial group theory brought a fresh perspective to the theory of groups by considering groups as geometric objects in their own right. Dehn functions of finite group presentations specifically has become a substantial area of study. The research done on the rigidity of groups in the 1960's and 70's has been deeply influential. The investigation of groups possessing varying levels of curvature- notably hyperbolic and relatively hyperbolic groups, have led to a flourishment of advances.

The analogues of low-dimensional topology

Philosophically speaking, the depth and beauty of 3-manifold theory is, it seems to me, mainly due to the fact that its theorems have offshoots that eventually blossom in a different subject, namely group theory.

John Stallings

There is an extensive connection between geometric group theory and low-dimensional topology- in particular, a connection between group splittings and embeddings of surfaces in 3-manifolds. Kneser's theorem (1929)[38] states that every compact orientable 3-manifold decomposes uniquely as a connected sum of 3-manifolds. Its group theoretic analogue, Grushko's theorem[26] was proven ten years later, stating that a finitely generated group that is the free product of two subgroups has rank equal to the sum of the ranks of these subgroups. Part of Knesner's proof in fact closely resembles Grushko's theorem.

In 1957, Papakyriakopoulos proved the sphere theorem[44] which provides conditions for elements of the second homotopy group of a 3-manifold to correspond to embedded spheres.
From 1968[50] to 1971[51] Stallings showed that a finitely generated group splits over a finite subgroup if and only if it has more than one end. This significant result connects the algebraic nature of a group with its geometric nature. Stallings noted that it was upon meditation of Papakyriakopoulos' sphere theorem that he came up with the method to prove this theorem[50] .

In 1979, Jaco and Shalen[32] developed independently from Johannson[31] , JSJ-decomposition, which takes an orientable Haken 3-manifold with incompressible boundary and canonically decomposes it by cutting along essential annuli and tori. This decomposition plays an important role in Perelman's proof of the geometrisation conjecture.

There exist JSJ-decomposition analogues for groups which were introduced by Kropholler for Poincaré duality groups[36] and by Sela[47] to answer questions about rigidity and the isomorphism problem for torsion-free hyperbolic groups. Generally speaking, it decomposes the group over a graph of groups, describing all possible decompositions as an amalgamated product or HNN-extension over subgroups that are elements of a given class. Sela showed that the JSJ decomposition of torsion-free one-ended word hyperbolic groups describes all its splittings over infinite cyclic subgroups and that it also encodes the outer automorphisms of the group. Sela used Rips' theory of group actions on $\mathbb{R}$-trees which shall be discussed in more detail. There are a number of different approaches with regards to JSJ-decomposition of groups.

There exists a dictionary between $SL_2(\mathbb{Z})$, the mapping class group $Mod(\Sigma)$ of a genus n surface Σ and the group of outer automorphisms of a free group $Out(F_n)$, with many of their respective properties finding an analogue. We have that $SL_2(\mathbb{Z})$ acts on $\mathbb{H}^2$. The mapping class group $Mod(\Sigma)$ acts properly discontinuously on Teichmüller space by composing the marking on the surface with a homeomorphism of the surface. The group $Out(F_n)$ acts on Culler and Vogtmann's Outer space, CV_n, also by changing the marking.

The ubiquitous nature of surface-groups and codimension-one subgroups

Waldhausen conjectured that the fundamental group of every closed irreducible 3-manifold with infinite fundamental group has a surface subgroup. Due to the geometrisation theorem, the final case that required a proof was that of hyperbolic 3-manifolds. This was proven by Kahn and Markovic in 2009[35] . This result played a fundamental role in Agol's proof of the Virtual Haken conjecture and Thurston's virtual fibering conjecture.

A natural generalisation of Waldhausen's conjecture is due to Gromov, asking if every one-ended word hyperbolic group contains a surface subgroup of genus at least two. The condition of one-endedness is to avoid trivial counterexamples such as finite groups, virtually free groups and free products of simpler hyperbolic groups. There are a number of results in the direction of this conjecture. Calegari and Walker[18] showed that a random group in the sense of Gromov contains a surface subgroup (and in fact, many) with probability approaching 1 as the the length n of the random relators goes to infinity.

Gordon, Long and Reid[23] showed that one-ended Coxeter groups contain hyperbolic surface subgroups. Calegari[12] developed his stable commutator length machinery and proved that if a group is the fundamental group of a graph of free groups with cyclic edge groups, is hyperbolic and has positive second Betti number, then it contains a surface subgroup. Dunwoody[21] , using his theory of minimal tracks on 2-complexes showed how to construct the Bass-Serre tree which G acts on using the subgroups H and a proper H-almost invariant set satisfying certain conditions.

A subspace Y of a geodesic metric space X is *quasiconvex* if there exists $k > 0$ such that every geodesic in X connecting $x, y \in Y$ lies in a k-neighbourhood of Y.
Let G be a finitely generated group with finite generating set S. A subgroup H of G is a *quasiconvex subgroup* if H is quasiconvex in the Cayley graph $\Gamma(G, S)$. By the Morse lemma, quasigeodesics in hyperbolic space remain within a bounded distance of a geodesic, and so quasiconvex subspaces share many properties with convex subspaces. Particularly in the case of hyperbolic groups, quasiconvex subgroups inherit useful characteristics. By the Morse lemma, quasiconvex subgroups of hyperbolic groups are hyperbolic. When $\partial G \approx \mathbb{S}^2$, quasiconvex surface subgroups are examples of codimension-one subgroups[37] .

Sageev[46] generalised Dunwoody's work using H-almost invariant subsets constructed from codimension-one subgroups H, and used them to construct a dual CAT(0)-cube complex upon which G acts. Sageev showed that given a finite collection of quasiconvex codimension-one subgroups of a group G then the action on the dual cube complex is cocompact, and that if G acts without a global fixed point on a CAT(0) cube complex then G has a codimension-one subgroup.

The relevance of codimension-one objects with respect to group splittings can be seen with Dunwoody's tracks, Sageev's codimension-one subgroups and also laminations on surfaces. Morgan and Shalen[39] showed that given a measured lamination on a surface, there is a dual action of the fundamental group of the surface on an $\mathbb{R}$-tree. Skora[49] showed that the small action of the fundamental group of a surface on a real tree is dual to a unique measured lamination on this surface. The *Rips machine*[3] is an algorithm which takes a band complex, which is a finite 2-complex with measured lamination and using a number of moves puts the 2-complex into a normal form, turning the lamination into a disjoint union of a finite number of sublaminations which are of four different "types", ultimately leading to Rips' theorem that if a finitely presented group acts freely by isometries on a real tree, then G is the free product of free abelian groups and closed surface groups.

A *Kleinian group* is a discrete subgroup of $SL_2(\mathbb{C})$, which can be thought of as a group of orientation preserving isometries of $\mathbb{H}^3$. Kleinian groups that do not possess torsion elements correspond to the fundamental groups of 3-manifolds. The quasi-conformal structure of the boundaries of hyperbolic spaces greatly informs their geometry and rigidity. Mostow first employed this in the proof of his celebrated *Mostow's rigidity theorem.* This notion extends to the boundaries of hyperbolic groups, which Cannon utilised to formulate the following important open conjecture, which is a group-theoretic generalisation of the generic case of Thurston's geometrisation conjecture for 3-manifolds.

Conjecture 1 (Cannon's conjecture)**.** *Suppose G is an infinite, finitely presented group whose Cayley graph is Gromov hyperbolic and whose space at infinity is the 2-sphere. Then G is a Kleinian group.*

Markovic[37] showed that the condition that any two points on the boundary $\partial G \approx \mathbb{S}^2$ can be separated by the limit set of a quasi-convex surface subgroup of a hyperbolic group G is equivalent to a positive solution to Cannon's conjecture, which roughly speaking means having "enough" quasiconvex surface subgroups allows G to be isomorphic to a Kleinian group. This important separation property was found to hold by Kahn and Markovic[35] in the case of cocompact Kleinian groups.

The nuances of negative and non-positive curvature

I have discovered such wonderful things that I was amazed, and it would be an everlasting piece of bad fortune if they were lost. When you, my dear Father, see them, you will understand; at present I can say nothing except this: that out of nothing I have created a strange new universe. All that I have sent you previously is like a house of cards in comparison with a tower. I am no less convinced that these discoveries will bring me honour than I would be if they were complete.

Janos Bolyai

Roughly speaking, the curvature of a space is a description of the extent and rate at which a space is expanding. The curvature of a space is instrumental in understanding its structure and behaviour. Spaces of non-positive and negative curvature are of great interest to geometric group theorists. There are a number of ways in which curvature is defined and consigned in geometric group theory.

CAT(0)-spaces

A space X is CAT(0) if every geodesic triangle T is "thinner" than its corresponding Euclidean comparison triangle. A metric space is *locally CAT(0)* if any point in the center of a CAT(0) ball of positive radius. For CAT(0)-spaces, just as it is the case for the sectional curvature of Riemannian manifolds-curvature is a local condition. In fact, Riemannian manifolds have non-positive sectional curvature if and only if they are locally CAT(0)-spaces (Theorem $\mathcal{C}$ II.1A.6.[7]). CAT(0)-spaces are unique geodesic spaces and are also contractible ($\mathcal{C}$ II.4,5[7]) . The landscape of CAT(0)-spaces is a wild terrain, but by adding conditions on such spaces to be proper, have extendible geodesics and be cocompact, CAT(0)-spaces become more approachable. Examples of CAT(0)-spaces include Euclidean buildings, Hilbert spaces, proper cocompact geodesically complete symmetric spaces (for instance $\mathbb{R}^n$ and $\mathbb{H}^n$), CAT(0)-polyhedral complexes (and in particular, CAT(0)-cube complexes), $\mathbb{R}$-trees (which are the 1-dimensional CAT(0)-spaces), and products of CAT(0)-spaces. By the Cartan-Hadamard theorem (C II.4.1[7]), further CAT(0)-spaces can be constructed by taking the universal cover of compact locally CAT(0) metric spaces. In the case of polyhedral complexes, Gromov's link condition is a combinatorial condition determining if the complex is CAT(0).

δ-hyperbolic spaces

Gromov introduced the notion of δ-hyperbolic spaces in the 1980's which generalised the metric of trees and classical hyperbolic geometry. A geodesic metric space is *δ-hyperbolic* if there exists a constant δ such that every geodesic triangle is *δ-thin*, that is if each side of the triangle lies within a δ-neighbourhood of the union of the other two sides. δ-hyperbolic spaces grow at an explosive rate and hyperbolicity is a quasi-isometric invariant. Finding such spaces is of great interest- one important example is that the curve-complex $\mathcal{C}(\Sigma)$ of a closed oriented genus n surface Σ is δ-hyperbolic. The outer-space analogue of the curve complex, the *complex of free factors* was also found to be hyperbolic[5] .

δ-hyperbolicity is a quasi-isometric invariant for geodesic metric spaces, where if $f : X \to Y$ is a (λ, μ)-quasi-isometry and Y is a δ-hyperbolic space, then X is a $\delta'(\delta, \lambda, \mu)$-hyperbolic space. δ-hyperbolicity is also equivalent to *geodesic stability*, that all quasi-geodesics segments are contained in a neighbourhood of some geodesic segment with the size of the neighbourhood only dependent on the constants associated to the quasi-geodesic. Examples of δ-hyperbolic spaces include $\mathbb{R}$-trees (with $\delta = 0$), hyperbolic polyhedral complexes (achieved by gluing hyperbolic polyhedra to satisfy Gromov's CAT(1) link condition), hyperbolic space $\mathbb{H}^n$ (with optimal $\delta = tanh^{-1}(\frac{1}{\sqrt{2}})$), Hadamard manifolds of negative sectional curvature (with $\delta = cosh^{-1}(\sqrt{2})$), complex hyperbolic space and symmetric spaces of rank-one with constant negative curvature. CAT(-1)-spaces more generally are also δ-hyperbolic. Hyperbolic groups in particular have been of great interest.

Hierarchically-hyperbolic spaces

Behrstock, Hagen and Sisto generalised the notion of a δ-hyperbolic space to that of *hierarchical hyperbolic space*[8] , initially to make a bridge between mapping class groups and cubical groups which was provoked by the strong analogues between the two classes of groups. There are many examples of groups which are not hyperbolic in the sense of Gromov that however in many respects exhibit hyperbolic behaviour, further motivating such a study. This class of spaces includes a large amount of cubical groups, mapping class groups, Teichmüller space, and hyperbolic spaces. Direct products of hierarchically hyperbolic spaces are also hierarchically hyperbolic, as are other arrangements of hierarchically hyperbolic spaces.

Properties that describe curvature

There are a number of characteristics of geodesic metric spaces that describe the spectrum of curvature of such a space. In the case of δ-hyperbolic spaces, all quasi-geodesics are Morse, strongly contracting, and have exponential divergence functions. *Isoperimetric inequalities* relate the length of a closed curve in a geodesic metric space with the infimal area of the discs that are bounded by that area. Geodesic metric spaces have a linear isoperimetric inequality if and only if they are δ-hyperbolic.
A *non-principal ultrafilter* on $\mathbb{N}$ is a finite additive probaility measure $\omega : \mathcal{P}(\mathbb{N}) \to \{0, 1\}$ such that $\omega(S) = 0$ is S is finite. For every bounded sequence a_n of real numbers, we define the *ultralimit* to be the unique number $l := lim_\omega a_n$ such that

$$\omega\{n : |a_n - l| < \epsilon\} = 1 \text{ for every } \epsilon > 0.$$

The *asymptotic cone* $Cone_\omega(X)$ of the metric space (X, d) for a non-principal ultrafilter ω and sequence of basepoints $(p_n) \in (X, \frac{d}{n})$ is defined to be the set of equivalence classes:

$$\frac{\left\{\text{sequences } (x_n) \in (X, \frac{d}{n}) \text{ with } \frac{d(x_n,p_n)}{n} \text{ bounded independentally of } n\right\}}{(x_n) \sim (y_n) \iff lim_\omega \frac{d(x_n,y_n)}{n} = 0}$$

equipped with the metric $d_\omega((x_n), (y_n)) = lim_\omega \frac{d(x_n,y_n)}{n}$. The asymptotic cone of a space can be thought of what is seen when viewing a space from increasingly distant observation points. The asymptotic cones of a CAT(0)-space are CAT(0), asymptotic cones of symmetric spaces are nondiscrete affine buildings and the asymptotic cones of δ-hyperbolic spaces are all $\mathbb{R}$-trees and so every point is a cut-point.

On the other side of the curvature spectrum, product spaces are neither Morse nor have strongly contracting geodesics and their divergence functions are linear. Every CAT(0)-space satisfies a quadratic isoperimetric inequality. The asymptotic cones of product spaces do not contain cut-points. Moreover, in the case of CAT(0)-spaces, Sultan[53] showed that quasi-geodesics being strongly contracting, Morse and having any two distinct points in the ultralimit separated by a cut-point in each asymptotic cone are equivalent.

Let X be a complete CAT(0)-space. A *contracting isometry* is a hyperbolic isometry having some axis (and hence any) λ such that the diameter of the orthogonal projection to λ is bounded above. A *rank-one isometry* is a hyperbolic isometry none of whose axes bound a half-plane. A contracting isometry is always a rank-one isometry. If X is proper, rank-one isometries are always contracting. The existence of contracting isometries is also indicative of hyperbolic behaviour- in particular in the case of a Gromov hyperbolic CAT(0)-spaces, every hyperbolic isometry is a contracting isometry.

The *visual boundary* of a geodesic metric space is the set of equivalence classes of geodesic rays, where two geodesics are equivalent if they remain within a bounded distance of one another. When two hyperbolic metric spaces are quasi-isometric, their visual boundaries are homeomorphic. This is not necessarily the case for CAT(0)-spaces, and so alternative descriptions of the boundary of such spaces were created that would be well-defined with respect to quasi-isometry for CAT(0)-spaces. The *contracting boundary* consists of equivalence classes of contracting rays (these are the rays that travel in a "hyperbolic" direction), under the same equivalence relation as the visual boundary. In the case of hyperbolic spaces, the contracting boundary recovers the visual boundary. The Morse boundary, similarly gives a well-defined notion of a boundary for a finitely generated group G.

Arboreal spaces

Bass-Serre theory, the study of groups acting on trees is a classical area of study first introduced by Serre (and developed further by Bass) in the 1970's[48] . This theory allows for the extraction of information of the structure of a group from its action on a tree, characterising free groups, amalgamated products and their generalisations as groups acting on simplicial trees.

There are many classes of spaces that behave similarly to trees. This is visible in echoing aspects of the structure of these spaces and the groups that act on them, as well as how trees are a particular subclass of many of these arboreal spaces.

CAT(0)-cube complexes

CAT(0)-cube complexes are tree-like spaces that are composed of cubes of varying dimensions that are isometrically glued together along their faces along with a combinatorial condition that the link of each vertex is flag. Trees are precisely the one-dimensional CAT(0)-cube complexes.

The study of CAT(0)-cube complexes has led to an abundance of results and implications with respect to groups and low-dimensional topology. One of the advantages of CAT(0)-cube complexes is that they contain hyperplanes- convex subsets $\mathfrak{h}$ that intersect the mid-point of edges of X such that $X \backslash \mathfrak{h}$ has two components. Hyperplanes entirely encode the structure of CAT(0)-cube complexes, with this encoding known as either its poc-set structure or wall-space.

A *wall space* is a set Y with a collection of non-empty subsets $\mathcal{H} \subset \mathcal{P}(Y)$ called *half-spaces* that are closed under the involution $* : \mathcal{H} \to \mathcal{H}$, $h \mapsto h^c = Y \backslash h$ and satisfy the *strong interval condition*:
For any $p, q \in Y$, there exists only finitely many half-spaces $h \in \mathcal{H}$ such that $p \in h$ and $q \in h^c$ (this set is denoted as $\mathcal{W}(x|y)$). The walls are the pairs $\overline{h} = \{h, h^c\}$ and the set of walls is $W = \mathcal{H}/*$. Two walls $\overline{h}$ and $\overline{k}$ *cross* if all four intersections $h \cap k$, $h \cap k^c$, $h^c \cap k$ and $h^c \cap k^c$ are nonempty.

Nica[40] showed that every wallspace has a canonical embedding into the 1-skeleton of a CAT(0)-cube complex and so a group action on a wall space extends to a group action on a CAT(0)-cube complex. Chatterji and Niblo[16] furthermore showed that a group acts (properly) isometrically on a wall space if and only if it acts (properly) isometrically on a CAT(0)-cube complex of dimension k, where k is equal to the number of pairwise crossing walls.

The extent to which the usefulness of hyperplanes has been exploited is eminently evident in Caprace and Sageev's *Rank Rigidity of CAT(0)-spaces*, which uses techniques that began emerging with Sageev's seminal PhD work. Sageev's PhD work generalised Stalling's theory of ends, showing that the number of *relative ends* $e(G, H)$ exceeds one if and only there exists an essential action of G on a CAT(0)-cube complex where the subgroup H stabilises a hyperplane. Sageev used the work of Scott that $e(G, H) > 1$ if and only if there exists a proper H-almost-invariant subset A of G such that $HA = A$.

Cubulation

The π_1-injective immersion of a manifold into a manifold of one higher dimension is a very natural construction which can be abstracted to the case of subgroups of codimension-one. For a finitely presented group G a subgroup H is a *codimension-one* subgroup of G if the number of ends of the coset graph C_G/H (denoted as $e(G, H)$) is 2 or greater. Sageev[46] showed that given a finitely generated hyperbolic group G and a quasi-convex subgroup H then either H is associated to a splitting of G, or H has a codimension-one subgroup. The H-almost invariant set is constructed by taking the complement of a neighbourhood N of H that separates the Cayley graph Γ of G into at least two components, with A taken to be one of these components.

Sageev constructed a dual CAT(0)-cube complex from a collection of codimension-one subgroups, where the vertex set $\mathcal{V}$ of the CAT(0)-cube complex consists of subsets of $\Sigma = \{gA | g \in G\} \cup \{gA^c | g \in G\}$ satisfying certain conditions that furthermore must be connected by an edge path to a vertex of the form $V_g = \{A \in \Sigma | g \in A\}$, where vertices are connected by an edge if their corresponding sets differ mutually by one element. Higher dimensional cubes are added inductively to the 1-skeleton, gluing in an n-cube whenever the boundary appears in the $(n-1)$-skeleton. Moreover, if G is a word hyperbolic group and $H_1, \ldots, H_k$ is a collection of quasiconvex codimension-one subgroups then the action of G on the dual cube complex is cocompact. The existence of codimension-one subgroups facilitates partition- as exhibited by the lifts of the curves on a surface to the universal cover, by Sageev's use of almost invariant sets and his introduction of hyperplanes. The cubulation of groups has lead to a cascade of progress, with Dani Wise especially understanding their importance and applicability.

A group G is *residually finite* if for every non-identity element $g \in G$ there exists a homomorphism $h : G \to K$ to a finite group K such that $h(g) \neq 1$. It is an important open conjecture of Gromov that every word-hyperbolic group is residually finite. The subject of residual finiteness has played an important role in Wise's work. As has subgroup separability, which coincides with the former class when the trivial subgroup of a group is subgroup separable. A subgroup H of G is *subgroup separable* if H is the intersection of finite index subgroups of G. Subgroup separability is a powerful tool that in the realm of topology, corresponds to immersions lifting to embeddings in a finite cover. Wise's construction of clean $\mathcal{VH}$-complexes was motivated by a pursuit of understanding subgroup separability. Wise then found that large classes of small-cancellation groups, which had long been conjectured to be residually finite, had codimension-one subgroups and so implemented Sageev's construction and began cubulating these groups amongst many others. Haglund and Wise[30] then generalised the "cleanliness" of $\mathcal{VH}$-complexes to the case of special cube complexes, which unlike $\mathcal{VH}$-complexes are of arbitrary dimension.

A non-positively curved cube complex is called *special* if every hyperplane is two-sided, no hyperplane self-intersects, no hyperplane directly self-osculates and no pair of hyperplanes inter-osculates. The fundamental groups of special cube complexes are precisely the subgroups of right-angled Artin groups. A cube complex is special if and only if it can be locally isometrically embedded into a Salvetti complex of a right-angled-Artin group. In particular, the fundamental groups of special cube complexes $\pi_1(X)$ are linear. This class of cube complexes satisfy the *canonical completion and retraction*, leading to the subgroup separability of quasiconvex subgroups. This lead Wise to formulate a program to understand groups- beginning by cubulating groups and then finding a special cover. In Wise's prolific work[56] , "The structure of groups with a quasiconvex hierarchy", he presented the following conjecture which he proved in the special case of hyperbolic groups with a quasiconvex hierarchy:

Conjecture 2. *A Gromov hyperbolic group that acts cocompactly on a* CAT(0)*-cube complex is the fundamental group of virtually special cube complexes.*

Agol proved this conjecture, which completed the journey towards solving Waldhausen's virtual Haken conjecture.

In the reverse direction, many results of the study of Hadamard manifolds have been extended to more general and neighbouring classes of spaces. *Hadamard manifolds* are complete and simply connected Riemannian manfolds with non-positive sectional curvature. One important example is the celebrated result of *rank-rigidity of Hadamard manifolds*[1] . It is in this context that the study of contracting and rank-one isometries emerged. Ballmann and Buyalo[2] formulated an analogous conjecture in the case of CAT(0)-spaces:

Conjecture 3 (Rank-rigidity conjecture for CAT(0)-spaces)**.** *Let X be a locally compact geodesically complete* CAT(0) *space and* Γ *be an infinite discrete group acting properly and cocompactly on X. If X is irreducible, then X is a higher rank symmetric space or a Euclidean building of dimension at least* 2*, or* Γ *contains a rank one isometry.*

Using the machinery of the combinatorics of hyperplanes, Caprace and Sageev[17] proved the rank-rigidity for CAT(0)-cube complexes:

Theorem 1. *Let X be a finite-dimensional* CAT(0)*-cube complex and* $\Gamma \leq \mathrm{Aut}(X)$ *be a group acting without fixed point in* $X \cup \partial_\infty X$*. Then there is a convex* Γ*-invariant subcomplex* $Y \subseteq X$ *such that either* Y *is a product of two unbounded cube subcomplexes or* Γ *contains an element acting on* Y *as a contracting isometry.*

Many important constructions and results are presented in Caprace and Sageev's *Rank rigidity for CAT(0)-cube complexes*. Finite-dimensional CAT(0)-cube complexes are shown to possess a canonical product decomposition $X = X_1 \times \cdots \times X_n$ of irreducible factors. This product decomposition corresponds to a partition $\sqcup_{i=1}^{n} \mathcal{H}_i$ of collections of hyperplanes of X such that all hyperplanes (which are disjoint from each other) in $\mathcal{H}_i$ cross all hyperplanes in $\mathcal{H}_j$ if $i \neq j$. Any group G acting on X has a subgroup of finite index that embeds in $\mathrm{Aut}(X_1) \times \cdots \times \mathrm{Aut}(X_n)$. Essential actions on CAT(0)-cube complexes are established as the indispensable approach to groups acting on CAT(0)-cube complexes. If a group of isometries $\Gamma \leq \mathrm{Aut}(X)$ acts on a finite-dimensional cube complex with either no fixed point at infinity or with finitely many orbis of hyperplanes, then the Γ-essential core is unbounded if and only if Γ acts non-trivially. The Γ-essential core then embeds as a Γ-invariant convex subcomplex of X. The *restriction quotient* of a cube complex is introduced, which creates new half-space systems by considering a subcollection of the hyperplanes of the original cube complex.

The relationship between the arrangements of hyperplanes and the behaviour of actions of CAT(0)-cube complexes is dissected in great detail- in particular, isometries are categorised and distinguished in terms of their behaviour with respect to hyperplanes. Isometries can flip, skewer, double-skewer, be parallel or peripheral to a hyperplane. The irreducibility of a CAT(0)-cube complex upon which its automorphism groups acts essentially and without a fixed point at infinity is shown to be equivalent to the existence of a pair of strongly separated hyperplanes. For $\mathrm{Aut}(X)$ acting essentially on a finite-dimensional CAT(0) cube complex X that either has no fixed point at infinity or acts acts cocompactly with X locally compact, the existence of an $\mathrm{Aut}(X)$-invariant flat in X is equivalent to X not containing a facing triple of hyperplanes. The existence of disjoint hyperplanes in opposing sectors of a pair of crossing hyperplanes in a finite-dimensional CAT(0)-cube complex with $\mathrm{Aut}(X)$ acting essentially and without a fixed point at infinity is proven. As a corollary of rigidity, Caprace and Sageev present a new proof of the Tits Alternative for groups acting on finite-dimensional CAT(0) cube complexes.

$\mathbb{R}$-trees

An $\mathbb{R}$-tree is a metric space $(\mathcal{T}, d)$ such that there is a unique geodesic between any two points $x, y \in \mathcal{T}$. $\mathbb{R}$-trees are precisely the class of 0-hyperbolic spaces- all geodesic triangles are tripods. Simplicial trees are basic examples of $\mathbb{R}$-trees, with much of their behaviour extending over to $\mathbb{R}$-trees. Isometric actions on $\mathbb{R}$-trees largely arise as the Gromov-Hausdorff limits of sequences of isometric actions of a group on negatively curved space. A group G acts *minimally* on an $\mathbb{R}$ if there is no G-invariant subset which is also an $\mathbb{R}$-tree. An isometric action of a group on an $\mathbb{R}$-tree is *small* if the pointwise stabilisers of non-degenerate arcs do not contain non-abelian free groups. The following theorem is due to Skora[49] :

Theorem 2. *If M is a closed hyperbolic surface, then any small action of* $\pi_1(M)$ *on an* $\mathbb{R}$*-tree is dual to a unique measured geodesic lamination on* M*.*

We note also that a measured lamination on a surface Σ is also dual to a point on Thurston's boundary of Teichmüller space. Skora's theorem is instrumental in Otal's proof of Thurston's double-limit theorem, which is important in the proof of the hyperbolisation theorem for 3-manifolds that fiber over the circle.

Morgan and Shalen[39] showed that all surface groups (with surface Euler characteristic less than -1) admit free actions on $\mathbb{R}$-trees. They also noted that the class of groups that act freely on $\mathbb{R}$-trees is closed under free products, which led them to ask if all finitely generated groups that act freely on $\mathbb{R}$-trees are free products of surface groups and finitely generated free abelian groups. Rips presented a proof at the Isle of Thorns conference in 1991 which answered this question in the affirmative:

Theorem 3. *If G is a finitely presented group that acts freely by isometries on an $\mathbb{R}$-tree, then G is the free product of free abelian groups and closed surface groups.*

Bestvina and Feighn[6] generalised the results of the Rips machine to the case of *stable actions*, which are actions that are non-trivial, minimal and every non-degenerate tree in the $\mathbb{R}$-tree has a stable substree (a tree is stable if all of its subtrees have the same pointwise stabilisers with respect to the action).
Guirardel[27] showed that any minimal stable action of a finitely presented group G of an $\mathbb{R}$-tree can be "approximated" by an action of G on a simplicial tree. Using this result as well as the Rips machinery, Guirardel also proved[28] that the minimal small action of one-ended hyperbolic groups on an $\mathbb{R}$-tree can be read from a JSJ decomposition of the group, with a corollary that any small action of of a one-ended hyperbolic group is dual to a measured foliation on a 2-complex, generalising Skora's duality result.
Sela used $\mathbb{R}$-trees to solve the isomorphism problem for torsion-free hyperbolic groups using the work of Rips as well constructing his JSJ decomposition for hyperbolic groups. $\mathbb{R}$-trees were also instrumental in his solution of the Tarski conjecture, showing that any two non-abelian finitely generated free groups possess the same first-order theory. $\mathbb{R}$-trees are also related to Culler and Vogtmann's outer-space.

Median spaces

A *median space* is a metric space X for which any three points $x_1, x_2, x_3 \in X$ have a *median point* m which means that $d(x_i, m) + d(m, x_j) = d(x_i, x_j)$ for all $1 \leq i < j \leq 3$. Examples of median spaces include $\mathbb{R}$-trees, CAT(0)-cube complexes with the ℓ^1-metric and $\mathbb{R}^n$ with the ℓ^1-metric. Products of median spaces are also median spaces, with the product given by the ℓ^1-metric. The *rank* of a median space X is the supremum over the set of integers k such that X contains an isometric copy of the cube $\{-a, a\}^k$ with $a > 0$ or equivalently the supremum over the cardinality of sets of pairwise transverse convex walls. $\mathbb{R}$-trees are the median spaces of rank one. The rank of a median s pace coincides with the dimension of CAT(0)-cube complexes. *Measured wall spaces* are an extension of the construction of wallspaces, which as a category is equivalent to the category of median spaces, demonstrated via isometric embeddings[13] .

The manner in which a group acts on a median space can be characterised by the Kazhdan and Haagerup properties[13] . A locally compact Hausdorff topological group G has *Kazhdan's property (T)* if for each unitary representation π of G, if π has almost invariant vectors, then it also has invariant vectors. In the opposing direction, G has the *Haagerup property* (or is *a-T-menable*) if it admits a proper action on a Hilbert space. Let G be a locally compact second countable group. The group G has property (T) if and only if any action by isometries on a median space has bounded orbits. A bounded orbit in this context implies the existence of a fixed point (Ch II, corollary 2.8(1))[7] . Examples of groups with property (T) include simple Lie groups of rank of at least two, in particular $SL_n(\mathbb{R})$ for $n \geq 3$ (and also its lattices). The group G has the Haagerup property if and only if it admits a proper action by isometries on a median space. Examples of groups with the Haagerup property include free groups, $SL_2(\mathbb{Z})$, $SL_2(\mathbb{Q}_p)$, Thompson's groups F, T, V, groups satisfying the $C'(\frac{1}{6})$-small cancellation condition and Coxeter groups.

Buildings

Buildings are combinatorial objects which generalise trees and projective spaces. The study of buildings can be seen as the converse of the Erlangen problem, as it is from group-theoretic constructions that buildings are created. Examples of such groups include simple Lie groups, algebraic groups over local fields, Kac-Moody groups and loop groups. The subgroups of these groups are also of notable interest, which includes uniform and irreducible lattices. By Margulis' arithmeticity theorem, irreducible lattices are mostly arithmetic and so arise from a number-theoretic construction. A group G with Coxeter system (W, S) associated to an action on a building has a *BN-pair structure*, which generalises the decomposition of general linear groups over a field. The *Borel subgroup* B, is the analogue of the group of upper-triangular matrices, the *Cartan subgroup* $H = B \cap N$ is the analogue of the group of diagonal matrices and N is the analogue of the normaliser of H. The group G admits a *Bruhat decomposition*

$G = \sqcup_{w \in W} BwB$.

When considering buildings as chamber systems admitting a Coxeter valued length function, an important theorem of Moussong states that this *Davis realisation* of an apartment is a CAT(0)-space[19] . This then can be extended to the entire geometric realisation of the building. Thus the techniques and results of the theory of CAT(0)-spaces can be applied to such buildings. Many Coxeter groups are generated by reflections in real hyperbolic spaces. The apartments of such buildings have negative curvature which then extends to the entire building. These associated buildings are called *hyperbolic buildings*. Examples include right-angled hyperbolic buildings (for instance-Bourdon's buildings $I_{p,q}$ in the 2-dimensional case) and hyperbolic triangular buildings. The 2-dimensional hyperbolic buildings are called *Fuchsian buildings*, with corresponding Coxeter groups discrete subgroups of $PGL_2(\mathbb{R})$. Further hyperbolic buildings can be constructed by using the amalgamation method of Tits and Haefliger[24] . The abundance of hyperbolic buildings presumably denies the possibility of their classification. Twin buildings Δ_+, Δ_- arise from Kac-Moody groups, induced by the twin BN-pairs, with the Kac-Moody group acting diagonally on the product of isomorphic buildings, $\Delta_+ \times \Delta_-$.

The interplay of the structure of both a space and the group that acts on it

> *Have a geometric space and some transformation group. A geometry is the study of those properties of the given geometric space that remain invariant under the transformations from this group. In other words, every geometry is the invariant theory of the given transformation group.*
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>
> Felix Klein, Vergleichende Betrachtungen über neuere geometrische Forschungen

> *To understand a group we need to see it in action. One may radically summarise the interest of Tits as providing geometrical structures on which groups are made to act – thus the reverse of the Erlangen programme. One of his more mature challenges was to give suggestive geometric interpretations to the exceptional Lie groups. Out of this, his elaborate theory with characteristic real estate terminology has evolved.*
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> Review by Ulf Persson of The Abel Prize 2008–2012, EMS Newsletter, March 2015

A common theme in geometric group theory is to find "natural" geometric realisations upon which a group acts, in order to understand the algebraic structure of the group. Nonabelian free groups acting on trees and semi-simple Lie groups acting on associated symmetric spaces are motivating examples of this approach. The work of Bruhat and Tits provides an abundance of further such examples. The fundamental example of such a natural geometric object is the Cayley graph, entirely encoding the structure of the group with respect to a choice of generating set. Much of the behaviour of groups is illuminated by the study of these group actions. Understanding the structure of the group has been an important motivation- the way a group splits for instance has been of great interest. Bass-Serre theory[48] gives the geometric interpretation of group splittings in the form: a finitely generated group G splits over a subgroup H if and only if G acts on a simplicial tree T with no global fixed points with edge orbit and edge stabiliser H. There have been a number of developments and generalisations due to Bass-Serre theory.

There are many classes of groups that have certain types of actions on families of spaces which have led to many promising results and research. One such class of groups are CAT(0)-groups. A group is called CAT(0) if it acts properly and cocompactly by isometries on a CAT(0)-space. CAT(0)-groups (and more generally, groups acting properly on CAT(0)-spaces) are closed under certain amalgamated free products ($\mathcal{C}$II.11.18[7]). CAT(0)-groups have many nice properties- a CAT(0)-group G satisfies:

- G is finitely presented.
- G has finitely many conjugacy classes of finite subgroups.

- All free abelian subgroups of G are finitely generated.
- G has solvable word and conjugacy problems.
- G has at most a quadratic Dehn function.

Another class of groups in this direction are hyperbolic groups- where a group is hyperbolic if and only if it admits a geometric action on a proper hyperbolic metric space.

Coarse geometry and quasi-isometry

> *...one may start to feel uncomfortable by realizing how much structure has been lost as one passed from G to the quasi-isometry class of G with its word metric. Indeed, one barters here the rigid crystalline beauty of a group for a soft and flabby chunk of geometry where all measurements have built-in errors. But something amazing and unexpected happens here as was discovered by Mostow in 1968: the quasi-isometric (or large-scale) geometry turns out to be far more rich and powerful than appears at first sight. In fact one believes nowadays that most essential elements of an infinite group are quasi-isometry invariant.*
>
> Mikhail Gromov

Mostow's rigidity theorem is a celebrated result of hyperbolic geometry that has led to many generalisations and applications. These include the superrigidity of non-uniform irreducible lattices in higher rank and results on arithmeticity by Margulis, as well as the development of hyperbolic groups by Gromov.

Theorem 4 (Mostow's Rigidity Theorem)**.** *If two closed manifolds of constant negative curvature and dimension at least* 3*, have isomorphism fundamental group, then they are isometric.*

Mostow's rigidity theorem says that the geometry of negatively curved manifolds of dimension above 2 are completely determined by their fundamental group and reveals hyperbolicity as the driving force behind this rigidity. The following is the corresponding Lie group form of this theorem:

Theorem 5. *Let* Γ *be a cocompact discrete subgroup of* $PSO(1,n)$*,* $n \geq 3$*. Suppose* $\phi : \Gamma \to PSO(1,n)$ *is a homomorphism whose image is again a cocompact discrete subgroup of* $PSO(1,n)$*. Then* ϕ *extends to an automorphism from* $PSO(1,n)$*.*

Given two metric spaces X and Y, we call a map $\phi : X \to Y$ a *quasi-isometry* if there are positive constants λ, μ such that for all $p, q \in X$:

$$\frac{1}{\lambda}d(x,y) - \mu \leq d(\phi x, \phi y) \leq \lambda d(x,y) + \mu$$

A quasi-isometry is distorting distances between metric spaces by a bounded factor above a fixed scale. The initial step in many proofs of Mostow's rigidity theorem involve showing that isomorphism between fundamental groups of the two manifolds determines a quasi-isometry of the corresponding universal covers, which leads to the corresponding boundary map being quasi-conformal. The importance of quasi-isometries is evident in the Milnor-Svarc lemma- a cornerstone of geometric group theory, uniting groups with spaces that act geometrically upon them.

Proposition 1 (Milnor-Svarc Lemma $\mathcal{C}$I.8.19[7])**.** *Let* X *be a metric space. If* Γ *acts properly and cocompactly by isometries on* X*, then* Γ *is finitely generated and for any choice of basepoint* $x_0 \in X$*, the map* $\gamma \mapsto \gamma \cdot x_0$ *is a quasi-isometry.*

A classic example of such a quasi-isometry is of a finitely generated group acting geometrically on its Cayley graph. Note that this quasi-isometry is independent of the choice of generating set for the group. The use of quasi-isometries in the work of Mostow in rigidity led to the interest in quasi-isometric rigidity, with Gromov notably seeking quasi-isometric classifications of groups. One motivating example was *Gromov's theorem on groups with polynomial growth*, which states that groups with polynomial growth are precisely the groups that have a nilpotent group of finite index- hence the class of nilpotent groups are isometrically rigid.

The study of quasi-isometries is an attempt to discern the degree to which they (an entirely geometric property) can command algebraic properties. Quasi-isometries have proven useful in low-dimensional topology as the fundamental group of a compact Riemannian manifold is quasi-isometric to the universal cover of the manifold. Many remarkable properties are preserved under quasi-isometry, with classes possessing such preserved roperties called *quasi-isometrically rigid*. Examples of quasi-isometrically rigid classes include (with an assumption of finite generation for the classes of groups):

- Free groups
- Free abelian groups
- Gromov-hyperbolicity is a quasi-isometric invariant in *geodesic metric spaces* (this is not the case in general metric spaces)
- Class of nilpotent groups
- Class of fundamental groups of closed surfaces
- Class of fundamental groups of closed 3-manifolds
- Class of finitely-presented groups
- Class of hyperbolic groups
- Class of amenable groups
- Class of fundamental groups of closed n-dimensional hyperbolic manifolds

In some cases, members of a class are all quasi-isometric- for instance, all regular simplicial trees are quasi-isometric to each other. Some examples of classes which are not quasi-isometrically rigid include:

- The property (T) is not a quasi-isometric invariant.
- The fundamental group of a compact non-positively curved Riemannian manifold with convex boundary is not a quasi-isometric invariant.
- Class of simple groups is not quasi-isometrically rigid.
- The class of residually-finite groups is not quasi-isometrically rigid (Burger-Mozes)

The geometric perspective and its influence

As long as algebra and geometry have been separated, their progress have been slow and their uses limited, but when these two sciences have been united, they have lent each mutual forces, and have marched together towards perfection.

Joseph Lagrange

Geometric group theory emerged in the late 19$^{\text{th}}$ century with the study of combinatorial group theory. The use of Cayley graphs to depict groups marked the beginning of viewing a group as a geometric object. Small cancellation theory saw the beginning of categorising and studying the curvature properties of groups from a combinatorial perspective. After the work of Dehn, the geometric study of groups lacked prominence up until Gromov's successful promotion of this approach. One important geometric aspect of a group is its curvature- which leads to many algebraic and combinatorial consequences. A group G is *hyperbolic* if there exists a generating set S (and hence for all generating sets) the associated Cayley graph $\Gamma(G,S)$ of G is hyperbolic.

Gromov introduced and developed the study of hyperbolic groups. Examples of hyperbolic groups include finite groups, finitely generated free groups, classes of small cancellation groups, as well as finite extensions and free products of finitely many word hyperbolic groups. Hyperbolic groups have finite presentation, admit a Dehn's algorithm and have a solvable isomorphism problem amongst many other properties.

The study of relatively hyperbolic groups was initiated by Gromov, motivated by the fundamental groups of hyperbolic manifolds with cusps, generalising the parabolic properties of the cusp subgroups. Farb also described a notion of relative hyperbolicity described by coning off cosets of the asociated Cayley graph of a group.
An isometric action of a group G on a metric space X is *acylindrical* if for every $\epsilon > 0$ there exist $R, N > 0$ such that for every two points $x, y \in X$ such that $d(x,y) \geq R$, there are at most N elements satisfying

$$d(x,gx) \leq \epsilon \text{ and } d(y,gy) \leq \epsilon.$$

A group G is *acylindrically hyperbolic* if it is not virtually cyclic and acts acylindrically on a hyperbolic space with unbounded orbits. Acylindrically hyperbolic groups include hyperbolic groups, relatively hyperbolic groups (except virtually cyclic groups), mapping class groups excluding mapping class groups of punctures closed surfaces and $Out(F_n)$.

Many classes of groups have powerful geometric interpretations. Braid groups are an important example- the potency and applicability of their geometric expression has become indispensable. Diagram groups are another example of groups with a predominantly geometric interpretation. *Coxeter groups* are a class of groups that also facilitate many geometric interpretations. The Thompson group F can be generated by a particular set of operations on binary trees which can be expressed in a particularly visual fashion.

In 1959, Jean Dieudonné, one of the founders of the Bourbaki group, yelled while at the Royaumont seminar- a French school education conference, “Down with Euclid! Death to triangles!” Dieudonné's article[20] for the conference was prefaced with the organisers describing his views as “extremist”. In the article he describes the Euclidean corpus as distant from modern mathematics, as not fundamental and “with no significance except as scattered relics of clumsy methods or an obsolete approach.”

The prevailing mathematical view at the time was one of formality and algebraic rigour, neglecting the geometric viewpoint. The Bourbaki group emphasised this with the absence of thorough publications in differential geometry and algebraic topology. Against this mathematical current, Donald Coxeter worked on polytopes and geometry- with his introduction of Coxeter groups a centerpiece in mathematics and an example of a beautiful bridge between the algebraic and geometric worlds, revealing just how intertwined these two modes of thought are.

In 1981, Robert Osserman- a geometer known for his work on minimal surfaces, describes in an essay[42] the apparent decline of geometry over the 20th century. He concludes prophetically by anticipating the prevalence of geometry in the mathematics to come, a number of years before the publication of Gromov's monumental *Hyperbolic groups*[25] with:

“I have said earlier that I believe that geometry has in fact suffered a serious decline. It has gone through a period of neglect, while the arbiters of mathematical taste and values were generally of the Bourbaki persuasion. On the other hand, I would say that there is nothing we need do about it, because that period is already drawing to a close. The recent rise to prominence of such unabashed geometers as Thurston and Yau is a sign that the low ebb in the fortunes of geometry has passed. I would predict that with no effort on any of our parts, we will witness a rebirth of geometry in the coming years, as the pendulum swings back from the extreme devotion to structure, abstraction, and generality.”

Open questions and directions

In the following, I wish to present some motivating open problems and directions that come from the literature.

Question 1. *What other classes of groups can be cubulated?*

Groups that have been cubulated include:

- Right-angled Artin groups[14]
- Coxeter groups[41]
- Finitely presented small cancellation groups[54]
- Hyperbolic free-by-cyclic groups
- Hyperoctahedral crystallographic groups[29]
- Tubular groups[55]
- Rhombus groups[33]
- Hyperbolic 3-manifold groups[10, 35]
- 1-relator groups with torsion
- Random groups of density $< \frac{1}{6}$ [43]

Question 2. *Does every hyperbolic group act properly and cocompactly by isometries on a* CAT(0)*-space? Or a* CAT(−1)*-space?*

Question 3. *Which Artin groups are* CAT(0)*?*[15]

Question 4. *Which Artin groups act properly on a* CAT(0)*-cube complex?*[15]

Question 5 (Bestvina)**.** *Are braid groups* CAT(0)*?*[34]

Question 6. *Characterise the right-angled Artin groups that contain closed surface subgroups*[9] *. Similarly: Is there an algorithm to decide for a given graph* Γ *whether the associated Artin group contains the fundamental group of a closed hyperbolic surface?*[15]

Question 7 (Delzant)**.** *Suppose that* G *acts isometric cocompact proper actions on two* CAT(0) *spaces* X_1 *and* X_2*. Is there a convex core for the diagonal action of* G *on* $X_1 \times X_2$*?*[34]

Question 8. *Can Rips' theory be extended to the product of* $\mathbb{R}$*-trees?*[34]

Question 9. *When does the mapping class group act on a* CAT(0)*-cube complex?*
Mapping class groups of surfaces of finite type cannot act properly by semi-simple isometries on any complete CAT(0)*-space (*$\mathcal{C}IIThm.7.26$[7]*).*

Question 10 (Genevois)**.** *Is a group which acts geometrically on a* CAT(0)*-cube complex and which does not contain* $\mathbb{Z}^2$ *as a subgroup Gromov hyperbolic?*[22]

Question 11 (Sageev, Wise)**.** *A group* G *satisfies the Tits alternative if for every subgroup* H *of* G*, either* H *is virtually solvable or* H *contains a free subgroup of rank 2.*
Does the Tits alternative hold for groups that act properly and cocompactly on a CAT(0)*-space?*[52]

Question 12 (Birman)**.** *Let* g, b *and* n *be the genus, number of boundary components and number of punctures respectively of a surface* S*.*
When is $MCG(S_{g,b,n})$ *linear?*
By Bigelow and Krammer, $MCG(S_{0,1,n}), MCG(S_{0,0,n})$ *and* $MCG(S_{2,0,0})$ *are linear.*

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