PREFERENCE BALLOT

A preference ballot is a ballot in which the voter ranks the choices in order of preference.

Example

A group of friends is deciding on a movie to watch for their monthly movie night. They have three options: Action (A), Comedy (C), and Drama (D). Here are their preferences:

	Alice	Bob	Lisa	Dave	Eric	Fiona	Greg	Hannah	Ian	Jessica
1st choice	A	A	C	D	C	C	D	D	A	A
2nd choice	C	C	A	C	D	D	A	A	C	C
3rd choice	D	D	D	A	A	A	C	C	D	D

	Action (A)	Comedy (C)	Drama (D)
1st choice	4	3	3
2nd choice	3	5	2
3rd choice	3	2	5

Plurality Method

In this method, the choice with the most first-preference votes is declared the winner.

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In this method, the choice with the most first-preference votes is declared the winner. Ties are possible, and would have to be settled through some sort of run-off vote.

	Alice	Bob	Lisa	Dave	Eric	Fiona	Greg	Hannah	Ian	Jessica
1st choice	A	A	C	D	C	C	D	D	A	A
2nd choice	C	C	A	C	D	D	A	A	C	C
3rd choice	D	D	D	A	A	A	C	C	D	D

	Action (A)	Comedy (C)	Drama (D)
1st choice	4	3	3
2nd choice	3	5	2
3rd choice	3	2	5

Action (A) won 4 out of 10 votes: It won in the plurality method but not majority

Question

In a student council election, three candidates are vying for the position of president: Alex (A), Brooke (B), and Chris (C). The voting schedule is provided below. Who wins under the plurality method?

	32	20	10	20	40
1st	A	B	C	C	A
2nd	B	C	B	A	C
3rd	C	A	A	B	B

	Alex (A)	Brooke (B)	Chris (C)
1st choice			
2nd choice			
3rd choice			

What's Wrong with Plurality?

	2	2	3	3
1st choice	A	A	O	H
2nd choice	O	H	H	A
3rd choice	H	O	A	O

Anaheim vs Orlando: 7 out of the 10 would prefer Anaheim over Orlando

Anaheim vs Hawaii: 6 out of 10 would prefer Hawaii over Anaheim

This doesn't seem right, does it? Anaheim just won the election, yet 6 out of 10 voters, 60% of them, would have preferred Hawaii!

FAIRNESS CRITERIA (Marquis de Condorcet)

The fairness criteria are statements that seem like they should be true in a fair election.

Condorcet Criterion

If there is a choice that is preferred in every one-to-one comparison with the other choices, that choice should be the winner. We call this winner the *Condorcet Winner*, or *Condorcet Candidate*.

Question

In a potluck party, attendees are voting for their preferred dish to be included in the menu. The options are Lasagna (LA), Tacos (TA), and Sushi (SU). Here's the preference schedule:

	1	3	3	3
1st choice	LA	LA	TA	SU
2nd choice	TA	SU	LA	TA
3rd choice	SU	TA	LA	LA

Lasagna (LA) vs Tacos (TA): _____ voters prefer
Lasagna (LA) vs Sushi (SU): _____ voters prefer
Tacos (TA) vs Sushi (SU): _____ voters prefer

_____ is the Condorcet winner

Example

Let's consider a scenario where a group of friends is voting for the destination of their next vacation. The options are Paris (PA), Rome (RO), and Tokyo (TO). Here's the preference schedule:

	1	3	3	3
1st choice	PA	PA	PA	TO
2nd choice	RO	TO	RO	PA
3rd choice	TO	RO	TO	RO

Paris (PA) vs Rome (RO): 10 out of 10 voters prefer Paris over Rome.
Paris (PA) vs Tokyo (TO): 7 out of 10 voters prefer Paris over Tokyo.
Rome (RO) vs Tokyo (TO): 4 out of 10 voters prefer Rome over Tokyo.
Based on these comparisons, Paris (PA) emerges as the Condorcet winner since it is preferred over both Rome and Tokyo in head-to-head matchups.

Example

Let's consider a university student government election in a campus with a diverse student body. In this election, there are three candidates: Sarah and Mike, both representing progressive ideologies, and Emily, a conservative candidate. The preference schedule for the votes looks as follows:

	375	245	234
1st choice	Emily	Sarah	Mike
2nd choice	Sarah	Mike	Sarah
3rd choice	Mike	Emily	Emily

We can see a total of 854 voters participated in this election. Computing the percentage of first-place votes: Sarah: 245/854 = 28.7%
Mike: 234/854 = 27.4% Emily: 375/854 = 43.9%

So in this election, the progressive voters split their votes between Sarah and Mike, allowing the conservative candidate Emily to win under the plurality method with 43.9% of the vote.

Example

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However, analyzing this election closer, we see that it violates the Condorcet Criterion. Analyzing the one-to-one comparisons:
Emily vs Sarah: 375 prefer Emily; 479 prefer Sarah: Sarah is preferred
Emily vs Mike: 375 prefer Emily; 479 prefer Mike: Mike is preferred
Sarah vs Mike: 620 prefer Sarah; 234 prefer Mike: Sarah is preferred
So even though Sarah had the smallest number of first-place votes in the election, she is the Condorcet winner, being preferred in every one-to-one comparison with the other candidates.

Question

Is there a Condorcet winner in the following?

	30	20	10	40	20	30
1st choice	A	A	B	C	C	B
2nd choice	B	C	C	B	A	A
3rd choice	C	B	A	A	B	C

Candidate A vs B:

Candidate A vs C:

Candidate B vs C:

Insincere voting

Situations when there are more than one candidate that share somewhat similar points of view, can lead to insincere voting. Insincere voting is when a person casts a ballot counter to their actual preference for strategic purposes.

Example of insincere voting

Imagine a fictional election for the Student Council President at a university. There are three candidates: Alice, Bob, and Claudia. Alice and Bob are both popular candidates and have similar platforms, while Claudia is less well-known and has different views.

A group of students strongly supports Alice but realizes that if they split their votes between Alice and Bob, Claudia might win.
To prevent Claudia from winning, some of Alice's supporters decide to strategically vote for Bob instead, even though they prefer Alice, to consolidate support behind one candidate. Similarly, some of Bob's supporters may also vote for Alice instead of Bob to ensure that Claudia doesn't win.

Instant Runoff Voting

Instant Runoff Voting (IRV), also called Plurality with Elimination, is a modification of the plurality method that attempts to address the issue of insincere voting. In IRV, voting is done with preference ballots, and a preference schedule is generated. The choice with the least first-place votes is then eliminated from the election, and any votes for that candidate are redistributed to the voters' next choice. This continues until a choice has a majority (over 50%).
(IRV can violate the Condercet Criterion)

Example of Instant Runoff Voting

	5	2	4	6	1	4
1st choice	B	C	B	D	B	E
2nd choice	C	A	D	C	E	A
3rd choice	A	D	C	A	A	D
4th choice	D	B	A	E	C	B
5th choice	E	E	E	B	D	C

There are a total of 22 voters. If this was a plurality election, B (with 10 out of 22) of first choice would win. So no one has a majority (12). Thus A is eliminated as A has no first place votes.

Example of Instant Runoff Voting

	5	2	4	6	1	4
1st choice	B	C	B	D	B	E
2nd choice	C	D	D	C	E	D
3rd choice	D	B	C	E	C	B
4th choice	E	E	E	B	D	C

We next eliminate C as C only has 2 first choice votes.

And the 2 first votes for C are distributed to D as D are these two voters' second choice.

Example of Instant Runoff Voting

	5	2	4	6	1	4
1st choice	B	D	B	D	B	E
2nd choice	D	B	D	E	E	D
3rd choice	E	E	E	B	D	B

	9	2	6	1	4
1st choice	B	D	D	B	E
2nd choice	D	B	E	E	D
3rd choice	E	E	B	D	B

Example of Instant Runoff Voting

	9	2	6	1	4
1st choice	B	D	D	B	E
2nd choice	D	B	E	E	D
3rd choice	E	E	B	D	B

We next eliminate E as it has 4 first choice votes

	9	2	6	1	4
1st choice	B	D	D	B	E
2nd choice	D	B			D
3rd choice			B	D	B

Example of Instant Runoff Voting

	9	2	6	1	4
1st choice	B	D	D	B	D
2nd choice	D	B	B	D	B

	10	12
1st choice	B	D
2nd choice	D	B

Thus D wins

Question

Number of voters	3	10	5	1	13	8	22
1st choice	W	W	C	C	D	X	W
2nd choice	X	C	W	X	X	C	D
3rd choice	C	D	X	D	W	D	C
4th choice	D	X	D	W	C	W	X

How many voters voted in this election?

How many first place votes are needed for a majority?

Which candidate/choice had the most first-place votes?

Which candidate/choice has the least first-place votes?

Which candidate/choice had the most last-place votes?

Which candidate/choice has the least last-place votes?

Question

Number of voters	8	13	12	7	12	7
1st choice	B	C	A	A	B	D
2nd choice	A	A	D	B	C	C
3rd choice	C	D	B	D	D	B
4th choice	D	B	C	C	A	A

Find the winner of this election under the plurality method.

Question

If there are 3 candidates in an election with a total of 25 votes, what is the minimum number of first-place votes a candidate could win with under the Plurality method?

MONOTONICITY CRITERION

If voters change their votes to increase the preference for a candidate, it should not harm that candidate's chances of winning.
(Improve or stay the same)

Suppose there are 3 candidates, and 100 votes cast.
The number of votes required to win is therefore 51.
Suppose the votes are cast as follows in an IRV election.

Number of ballots	1st Preference	2nd Preference
39	Andrew	Belinda
35	Belinda	Carly
26	Carly	Andrew

Carly is eliminated as Carly has 26 first place votes.

Number of ballots	1st Preference	2nd Preference
39	Andrew	Belinda
35	Belinda	
26		Andrew

So Andrew wins (39+26=65) to Belinda's 35

Suppose there are 3 candidates, and 100 votes cast.
The number of votes required to win is therefore 51.
Suppose the votes are cast as follows in an IRV election.

Number of ballots	1st Preference	2nd Preference
39	Andrew	Belinda
35	Belinda	Carly
26	Carly	Andrew

Now suppose 10 Belinda voters drop their support for her and rank Andrew first instead.

Number of ballots	1st Preference	2nd Preference
49	Andrew	Belinda
25	Belinda	Carly
26	Carly	Andrew

Now suppose 10 Belinda voters drop their support for her and rank Andrew first instead.

Number of ballots	1st Preference	2nd Preference
49	Andrew	Belinda
25	Belinda	Carly
26	Carly	Andrew

Belinda has the least first voters and is eliminated

Number of ballots	1st Preference	2nd Preference
49	Andrew	
25		Carly
26	Carly	Andrew

This time Carly now has 51 votes and wins over Andrew, despite Andrew receiving 10 of Belinda's votes.

Borda Count

In this method, points are assigned to candidates based on their ranking;
1 point for last choice,
2 points for second-to-last choice,
and so on. The point values for all ballots are totaled, and the candidate with the largest point total is the winner.

Example of Borda Count

	9	2	6	1	4
1st choice	B	D	D	B	E
2nd choice	D	B	E	E	D
3rd choice	E	E	B	D	B

	9	2	6	1	4
1st choice	B	D	D	B	E
	3·9	3·2	3·6	3·1	3·4
2nd choice	D	B	E	E	D
	2·9	2·2	2·6	2·1	2·4
3rd choice	E	E	B	D	B
	1·9	1·2	1·6	1·1	1·4

Example of Borda Count

	9	2	6	1	4
1st choice	B	D	D	B	E
	27	6	18	3	12
2nd choice	D	B	E	E	D
	18	4	12	2	8
3rd choice	E	E	B	D	B
	9	2	6	1	4

B: 27 + 4 + 6 + 3 + 4 = 44

D: 18 + 6 + 18 + 1 + 8 = 51

E: 9 + 2 + 12 + 2 + 12 = 37

Under the Borda count method, D is the winner.

Note that B had more first choice voters over D but lost. In the Borda count, it can even be the case that a choice with a majority of first place counts loses.

MAJORITY CRITERION

If a choice has a majority of first-place votes, that choice should be the winner.

Copeland's Method

In this method, each pair of candidates is compared, using all preferences to determine which of the two is more preferred. The more preferred candidate is awarded 1 point. If there is a tie, each candidate is awarded ½ point. After all pairwise comparisons are made, the candidate with the most points, and hence the most pairwise wins, is declared the winner.

The Copeland method satisfies the Condercet Criterion, Majority Criterion and Monotonicity Criterion.

Example

Number of voters	8	13	12	7	12	7
1st choice	B	C	A	A	B	D
2nd choice	A	A	D	B	C	C
3rd choice	C	D	B	D	D	B
4th choice	D	B	C	C	A	A

A vs B : A=13+12+7= 32 B=8+12+7=27 : A 1 point

A vs C : A= 8+12+7= 27 C=13+12+7=32 : C 1 point

B vs C: B=8+12+7+12= 39 C=13+7=20 : B 1 point

B vs D: B=8+7+12= 27 D=13+12+7= 32: D 1 point

C vs D: C=8+13+12= 33 D=12+7+7= 26 : C 1 point

A vs D: A=8+13+12+7=40 D=12+7= 19 : A 1 point

A:2 points, B:1 point, C: 2 points, D: 1 point, so A & D tie

Example

In a school election for student council president, four candidates are competing: Alex (A), Beth (B), Chris (C), and Diana (D). The votes are cast as follows:

	5	4	6	3
1st choice	A	B	C	D
2nd choice	B	C	D	A
3rd choice	C	D	A	B
4th choice	D	A	B	C

A vs B: 14 votes to 4 votes, A gets 1 point
A vs C: 8 votes to 10 votes, C gets 1 point
A vs D: 5 votes to 13 votes, D gets 1 point
B vs C: 12 votes to 6 votes, B gets 1 point
B vs D: 9 votes to 9 votes, B & D get 1/2 point
C vs D: 15 votes to 3 votes, C gets 1 point

Totaling:
A has 1 point
B has 1 ½ points
C has 2 points
D has 1 ½ points

So C wins

Example

Candidate A is then removed from the election:

	5	4	6	3
1st choice	A	B	C	D
2nd choice	B	C	D	A
3rd choice	C	D	A	B
4th choice	D	A	B	C

	5	4	6	3
1st choice		B	C	D
2nd choice	B	C	D	
3rd choice	C	D		B
4th choice	D		B	C

	5	4	6	3
1st choice	B	B	C	D
2nd choice	C	C	D	B
3rd choice	D	D	B	C

B vs C: 13 votes to 5 votes, B gets 1 point

B vs D: 9 votes to 9 votes, B & D get 1/2 point

C vs D: 15 votes to 3 votes, C gets 1 point

Totaling:

B has 1 ½ points

C has 1 point

D has ½ point

So B wins

THE INDEPENDENCE OF IRRELEVANT ALTERNATIVES (IIA) CRITERION

If a non-winning choice is removed from the ballot, it should not change the winner of the election.

Equivalently, if choice A is preferred over choice B, introducing or removing a choice C should not cause B to be preferred over A.

So Where's the Fair Method?

At this point, you're probably asking why we keep looking at method after method just to point out that they are not fully fair. We must be holding out on the perfect method, right?

Unfortunately, no. A mathematical economist, Kenneth Arrow, was able to prove in 1949 that there is no voting method that will satisfy all the fairness criteria we have discussed.

Arrow's Impossibility Theorem

Arrow's Impossibility Theorem states, roughly, that it is not possible for a voting method to satisfy every fairness criteria that we've discussed.

Example

10 people prefer A to B
10 people prefer B to C
10 people prefer C to A

Consider the election below:

	5	5	5
1st choice	A	C	B
2nd choice	B	A	C
3rd choice	C	B	A

No matter whom we choose as the winner, 2/3 of voters would prefer someone else! This scenario is dubbed Condorcet's Voting Paradox, and demonstrates how voting preferences are not transitive (just because A is preferred over B, and B over C, does not mean A is preferred over C). In this election, there is no fair resolution.

It is because of this impossibility of a totally fair method that Plurality, IRV, Borda Count, Copeland's Method, and dozens of variants are all still used. Usually the decision of which method to use is based on what seems most fair for the situation in which it is being applied.

Approval Voting

With Approval Voting, the ballot asks you to mark all choices that you find acceptable. The results are tallied, and the option with the most approval is the winner.

Approval Voting

A group of friends is deciding on which restaurant to go to for dinner. Three options are provided, and each person is asked to mark with an "X" which restaurants they are willing to go to. The results are:

	Alex	Ben	Cara	Dan	Emma	Frank	Gina	Hannah	Ian	Julia
Italian	X	X	X	X	X		X		X	
Sushi		X		X	X		X		X	X
Mexican	X		X	X	X	X	X		X	X

Totaling the results, we find:
Italian received 7 approvals, Sushi received 6 approvals
Mexican received 8 approvals.
In this vote, Mexican would be the winner.

Example

	80	15	5
1st choice	A	B	C
2nd choice	B	C	B
3rd choice	C	A	A

Suppose that this election was held using Approval Voting, and every voter marked approval of their top two candidates. A is the winner

We will change this to approval voting where the top 2 choices are approved.

	80	15	5
A	X		
B	X	X	X
C		X	X

A has 80 votes, B has 100 votes and C has 20 votes,
so B is the winner

Approval Voting

	Mila	AJ	Aisha	Diego	Mei	Sanjay	Fatima	Amir	Ling	Natasha
A	X		X			X		X		X
B		X	X		X	X	X		X	
C	X	X	X	X	X		X	X		

A received 5 approvals, B received 6 approvals and C received 7. So C is the winner. Suppose AJ and Mei remove C from their approvals to try to get B win.

	Mila	AJ	Aisha	Diego	Mei	Sanjay	Fatima	Amir	Ling	Natasha
A	X		X			X		X		X
B		X	X		X	X	X		X	
C	X		X	X			X	X		

Now A has 5 approvals, B has 6 approvals and C received 5. So B is now the winner.

Question

Use the Copeland method to find the winner of the following:

Number of Voters	10	8	14	9
1st Choice	B	C	A	D
2nd Choice	A	D	C	B
3rd Choice	C	A	D	C
4th Choice	D	B	B	A

Question

Find the Condercet Candidate if they exist:

Number of voters	6	9	12
1st choice	B	B	A
2nd choice	C	A	B
3rd choice	A	C	C

Question

Find the winner under the IRV method:

Number of voters	10	3	7	2	9
1st choice	D	A	C	B	A
2nd choice	B	B	B	C	C
3rd choice	C	D	D	A	B
4th choice	A	C	A	D	D

Question

Find the winner under the Borda Count:

Number of voters	15	6	12	6	2
1st choice	C	D	A	B	D
2nd choice	A	A	D	D	B
3rd choice	B	C	B	A	C
4th choice	D	B	C	C	A

Question

What Fairness Criterion is violated in the following Borda count election?

		Number of voters		
Borda points	Rankings	6	3	2
3	1st choice	F	S	M
2	2nd choice	S	M	S
1	3rd choice	M	F	F

F:6(3)+3(1)+2(1)=23
S:6(2)+3(3)+2(2)=25
M:6(1)+3(2)+2(3)=18

A Majority Criterion
B Monotonicity Criterion
C Independence of Irrelevant Alternatives Criterion
D Condorcet Criterion

Question

A city is voting for its next mayor, with three candidates: Candidate A, Candidate B, and Candidate C. The winning candidate under the unknown voting method is Candidate A.

However, a significant portion of the population raises concerns about Candidate B's integrity, citing recent scandals. To address these concerns, they
propose removing Candidate B from the options and holding a revote with only Candidate A and Candidate C. In the second round of voting, Candidate C emerges as the winner. Which fairness criterion has been violated in this election?

A Majority Criterion
B Monotonicity Criterion
C Independence of Irrelevant Alternatives Criterion
D Condorcet Criterion